TEMPERATURE AND THERMAL STRESS DISTRIBUTION IN A CYLINDRICAL SHELL FILLED WITH LIQUID

G. N. Zamula

Inzhenerno-Fizicheskii Zhurnal, Vol. 9, No. 5, pp. 667–673, 1965

UDC 536.248+536.3

An approximate solution is given for the problem of steady-state temperature and thermal-stress distribution in a thin cylindrical shell partially filled with a liquid. Radiative heat transfer is taken into account.

Consider the steady-state convective heating of a thin vertical cylindrical shell partially filled with liquid to a constant or slowly varying level. Let the temperature of the outer diathermic surroundings be T_e, and assume that the film coefficients between the surroundings and the shell h and between the shell and the liquid h_l, as well as the physical properties of the shell, are uniform and independent of temperature. The shell is assumed to be gray with respect to radiation and sufficiently long for end effects to be negligible. The liquid is black with respect to radiation and has a constant temperature T_l. Under these assumptions, taking account of the internal radiative heat transfer in the shell, we obtain the heat-balance equations for the shell:

$$\left(\frac{\delta}{D}\right)^2 \frac{1}{\mathrm{Bi}} \frac{d^2 t_1(\xi)}{d\xi^2} + 1 - t_1(\xi) - \sigma t_1^4(\xi) - \sigma q(\xi) = 0, \quad \xi > 0, \tag{1}$$

$$\left(\frac{\delta}{D}\right)^2 \frac{1}{\mathrm{Bi}} \frac{d^2 t_2(\xi)}{d\xi^2} + 1 - t_2(\xi) - m\left[t_2(\xi) - t_l\right] - \sigma t_2^4(\xi) = 0, \quad \xi < 0, \tag{2}$$

$$q(\xi) = t_1^4(\xi) - \int_0^\infty t_1^4(\eta) K(\xi, \eta)\, d\eta - t_l^4 \int_{-\infty}^0 K(\xi, \eta)\, d\eta + (1-\varepsilon)\int_0^\infty q(\eta) K(\xi, \eta)\, d\eta, \quad \xi > 0, \tag{3}$$

with the boundary conditions

$$t_1(0) = t_2(0), \quad \frac{dt_1}{d\xi}(0) = \frac{dt_2}{d\xi}(0),$$

$$\frac{dt_1}{d\xi}(\infty) = 0, \quad \frac{dt_2}{d\xi}(-\infty) = 0, \tag{4}$$

where

$$\xi = x/D, \quad t = T/T_e, \quad q = q_{\mathrm{net}}/\varepsilon c_0 T_e^4, \quad t_l = T_l/T_e,$$

$$m = h_l/h, \quad \mathrm{Bi} = h\delta/\lambda, \quad \sigma = \varepsilon c_0 T_e^3/h.$$

The integral equation of radiative transfer (3) is a Fredholm equation of the second kind with a symmetric kernel [1], which, in the case of a circular cylinder, can be represented in the form [2, 3]

$$K(\xi, \eta) = 1 - [(\xi-\eta)^2 + 3/2]\,|\xi-\eta|/[(\xi-\eta)^2 + 1]^{3/2}, \tag{5}$$

yielding for $\xi > \eta$

$$\int_{-\infty}^{\eta} K(\xi, \eta)\, d\eta = \frac{(\xi-\eta)^2 + 1/2}{[(\xi-\eta)^2+1]^{1/2}} - (\xi - \eta). \tag{6}$$

It is known [3], that the kernel (5) of the integral equation can be approximated with sufficient accuracy by the function $\exp(-2|\xi - \eta|)$,

$$K(\xi, \eta) \simeq \exp(-2|\xi-\eta|), \tag{5'}$$

so that the integral (6) becomes

$$\int_{-\infty}^{\eta} K(\xi, \eta)\, d\eta \simeq \int_{-\infty}^{\eta} \exp[-2(\xi-\eta)]\, d\eta = \frac{1}{2}\exp[-2(\xi-\eta)] \quad (\xi > \eta). \tag{6'}$$

Replacing (5) and (6) in equation (3) by their approximations (5') and (6'), we obtain

$$q(\xi) = t_1^4(\xi) - \int_0^\xi t_1^4(\eta)\exp[-2(\xi-\eta)]\, d\eta - \int_\xi^\infty t_1^4(\eta)\exp[-2(\eta-\xi)]\, d\eta - \frac{t_l^4}{2}\exp(-2\xi) + (1-\varepsilon)\int_0^\xi q(\eta)\exp[-2(\xi-\eta)]\, d\eta + (1-\varepsilon)\int_\xi^\infty q(\eta)\exp[-2(\eta-\xi)]\, d\eta. \tag{3'}$$

Differentiating (3') twice and subtracting $4q(\xi)$ from $d^2q(\xi)/d\xi^2$, we obtain the differential equation for $q(\xi)$,

$$\frac{d^2 q(\xi)}{d\xi^2} - 4\varepsilon q(\xi) = \frac{d^2[t_1^4(\xi)]}{d\xi^2}, \quad \xi > 0. \tag{7}$$

The differential equation (7) is equivalent to the integral equation (3') and can be used as an approximation to equation (3). The boundary conditions for (7) can be obtained by satisfying the integral equation (3') at two points. At $\xi = 0$ we have

$$q(0) = t_1^4(0) - \frac{t_l^4}{2} - \int_0^\infty t_1^4(\eta)\exp(-2\eta)\,d\eta +$$

$$+ (1-\varepsilon)\int_0^\infty q(\eta)\exp(-2\eta)\,d\eta, \qquad (8)$$

and at $\xi \to \infty$ we have, in view of (3') and (4'),

$$q(\infty) = 0. \qquad (9)$$

In the special case when $t_1 = \Theta = \text{const}$ ($\Theta = T/T_e$), the solution of (7) with the boundary conditions (8), (9) is

$$q(\xi) = \frac{1}{1+\sqrt{\varepsilon}}(\Theta^4 - t_l^4)\exp(-2\sqrt{\varepsilon}\xi). \qquad (10)$$

This solution holds in the case of a semi-infinite uniformly heated cylindrical shell radiating into space ($T_l = 0$) or toward a perfectly black bottom with the temperature T_l. We see that the "edge" of the shell ($\xi = 0$) loses the net radiative heat flux

$$q_{\text{net}}(0) = \varepsilon c_0 T_e^4 \cdot q(0) = \frac{\varepsilon c_0}{1+\sqrt{\varepsilon}}(T^4 - T_l^4),$$

and the attenuation length of $q_{\text{net}}(x)$ is approximately $D/\sqrt{\varepsilon}$.

Consider now the system of differential equations (1), (2), (7) with the boundary conditions (4), (8), (9). It can be easily seen that in the absence of radiation ($\sigma = 0$) the solution is

$$t_1(\xi) = \Theta_1 - \vartheta_1\exp(-\gamma_1\xi), \quad \xi > 0,$$

$$t_2(\xi) = \Theta_2 + \vartheta_2\exp(\gamma_2\xi), \quad \xi < 0, \qquad (11)$$

where

$$\Theta_1 = 1, \ \vartheta_1 = \frac{\sqrt{1+m}}{1+\sqrt{1+m}}(\Theta_1 - \Theta_2), \ \gamma_1 = \frac{D}{\delta}\sqrt{\text{Bi}},$$

$$\Theta_2 = \frac{1+mt_l}{1+m}, \ \vartheta_2 = \frac{1}{1+\sqrt{1+m}}(\Theta_1 - \Theta_2),$$

$$\gamma_2 = \frac{D}{\delta}\sqrt{\text{Bi}(1+m)}.$$

Using the collocation method, we shall seek an approximate solution of the system (1), (2) in the form (11), where Θ_1, Θ_2 are the steady-state temperatures at $+\infty$, and $-\infty$, given by

$$1 - \Theta_1 - \sigma\Theta_1^4 = 0, \ 1 + mt_l - (1+m)\Theta_2 - \sigma\Theta_2^4 = 0. \qquad (12)$$

The constants ϑ_1, ϑ_2, γ_1, γ_2 are determed by the two compatability conditions at $\xi = 0$ (4), which yield

$$\vartheta_1 + \vartheta_2 = \Theta_1 - \Theta_2, \quad \gamma_1\vartheta_1 = \gamma_2\vartheta_2, \qquad (13)$$

and the two conditions requiring that equations (1), (2) be satisfied at $\xi = 0$. To write down the latter conditions in explicit form, we must first solve equation (7) for $t_1(\xi)$ as given by (11). Taking account of (9) and assuming $n^2\gamma_1^2 \neq 4\varepsilon$, n = 1, 2, 3, 4, we have

$$q(\xi) = C\exp(-2\sqrt{\varepsilon}\xi) - 4\gamma_1^2\vartheta_1\left[\frac{\Theta_1^3}{\gamma_1^2 - 4\varepsilon}\exp(-\gamma_1\xi) - \right.$$

$$- \frac{6\Theta_1^2\vartheta_1}{4\gamma_1^2 - 4\varepsilon}\exp(-2\gamma_1\xi) + \frac{9\Theta_1\vartheta_1^2}{9\gamma_1^2 - 4\varepsilon} \times$$

$$\left. \times \exp(-3\gamma_1\xi) - \frac{4\vartheta_1^3}{16\gamma_1^2 - 4\varepsilon}\exp(-4\gamma_1\xi)\right], \ \xi > 0, \qquad (14)$$

where C is determined by (8),

$$C = \frac{1}{1+\sqrt{\varepsilon}}(\Theta_1^4 - t_l^4) + \frac{2\varepsilon\vartheta_1}{1+\sqrt{\varepsilon}}\left[\frac{4(2+\gamma_1)}{\gamma_1^2 - 4\varepsilon}\Theta_1^3 - \right.$$

$$- \frac{6(2+2\gamma_1)}{4\gamma_1^2 - 4\varepsilon}\Theta_1^2\vartheta_1 + \frac{4(2+3\gamma_1)}{4\gamma_1^2 - 4\varepsilon}\Theta_1\vartheta_1^2 -$$

$$\left. - \frac{2+4\gamma_1}{16\gamma_1^2 - 4\varepsilon}\vartheta_1^3\right]. \qquad (15)$$

Substituting $t(\xi)$ from (11) and $q(\xi)$ from (14) into equations (1) and (2) for $\xi = 0$, we obtain the nonlinear algebraic equations for the constants ϑ_1, ϑ_2, γ_1, γ_2.

$$2\vartheta_1^4\sigma\left[1 - \frac{\varepsilon}{1+\sqrt{\varepsilon}}\frac{1}{4\gamma_1 + 2\sqrt{\varepsilon}}\right] -$$

$$- 8\vartheta_1^3\Theta_1\sigma\left[1 - \frac{\varepsilon}{1+\sqrt{\varepsilon}}\frac{1}{3\gamma_1 + 2\sqrt{\varepsilon}}\right] +$$

$$+ 12\vartheta_1^2\Theta_1^2\sigma\left[1 - \frac{\varepsilon}{1+\sqrt{\varepsilon}}\frac{1}{2\gamma_1 + 2\sqrt{\varepsilon}}\right] -$$

$$- \vartheta_1\left[8\Theta_1^3\sigma\left(1 - \frac{\varepsilon}{1+\sqrt{\varepsilon}}\frac{1}{\gamma_1 + 2\sqrt{\varepsilon}}\right) + 1 - \right.$$

$$\left. - \left(\frac{\delta}{D}\right)^2\frac{\gamma_1^2}{\text{Bi}}\right] + \frac{\sigma}{1+\sqrt{\varepsilon}}(\Theta_1^4 - t_l^4) = 0,$$

$$\sigma\vartheta_2^3 + 4\sigma\vartheta_2^2\Theta_2 + 6\sigma\vartheta_2\Theta_2^2 + 4\sigma\Theta_2^3 +$$

$$+ \ 1 + m - \left(\frac{\delta}{D}\right)^2\frac{\gamma_2^2}{\text{Bi}} = 0. \qquad (16)$$

Thus, with Θ_1 and Θ_2 determined by (12), we have 4 algebraic equations (13), (16) for ϑ_1, ϑ_2, γ_1, γ_2. Solving these equations numerically (e.g. by successive approximations), we obtain an approximate temperature profile in the shell in the form of equation (11), which for $\sigma = 0$ reduces to the exact radiationless solution.

In the case $\gamma_1, \gamma_2 >> 1$, which is the most interesting one from the practical point of view, equations (16) yield

$$\gamma_1^2 = \left(\frac{D}{\delta}\right)^2 \mathrm{Bi}\left[1 + 2\sigma\,\frac{\Theta_1^4-(\Theta_1-\vartheta_1)^4-\frac{1}{2(1+\sqrt{\varepsilon})}(\Theta_1^4-t_l^4)}{\vartheta_1}\right] = f_1(\vartheta_1),$$

$$\gamma_2^2 = \left(\frac{D}{\delta}\right)^2 \mathrm{Bi}\left[1+m+\sigma\,\frac{(\Theta_2+\vartheta_2)^4-\Theta_2^4}{\vartheta_2}\right] = f_2(\vartheta_2). \quad (17)$$

Since, in view of (13),

$$\vartheta_2 = \Theta_1 - \Theta_2 - \vartheta_1, \quad \gamma_2 = \gamma_1\,\frac{\vartheta_1}{\Theta_1-\Theta_2-\vartheta_1}, \quad (18)$$

the second equation in (17) yields

$$\gamma_1^2 = \left[\frac{\Theta_1-\Theta_2-\vartheta_1}{\vartheta_1}\right]^2 f_2(\Theta_1-\Theta_2-\vartheta_1) = \varphi_1(\vartheta_1).$$

The intersection of the two curves $\gamma_1^2 = f_1(\vartheta_1)$ and $\gamma_1^2 = \varphi_1(\vartheta_1)$, $0 < \vartheta_1 < \Theta_1 - \Theta_2$ determines γ_1, ϑ_1, and hence, with (18), γ_2, ϑ_2.

In the case when the heat transfer on the liquid side predominates, $m \gg 1$, (12), (17), and (18) can be approximated by

$$\Theta_2 = t_l, \quad \vartheta_2 = 0, \quad \vartheta_1 = \Theta_1 - t_l,$$

$$\gamma_1^2 = \left(\frac{D}{\delta}\right)^2 \mathrm{Bi}\left\{1+2\sigma\left[1-\frac{1}{2(1+\sqrt{\varepsilon})}\right](\Theta_1^2+t_l^2)(\Theta_1+t_l)\right\}. \quad (19)$$

and the temperature profile in the shell is then

$$t_1(\xi) = \Theta_1 - (\Theta_1 - t_l)\exp(-\gamma_1\xi), \quad \xi > 0,$$
$$t_2(\xi) = t_l, \quad \xi < 0, \quad (20)$$

where Θ_1 is determined by the first equation in (12).

Using the temperature profile (11), we can find the steady-state thermal stresses produced in the shell by the nonuniform heating. The deflection of a thin cylindrical shell in an axisymmetric temperature field $U(\xi) = 2\omega(\xi)/D\alpha T_e$ is determined within an additive constant by the equation [4]

$$\frac{d^4U(\xi)}{d\xi^4} + 4k^4U(\xi) = 4k^4t(\xi),$$

$$k^4 = 12(1-\nu^2)\left(\frac{D}{\delta}\right)^2 \quad (21)$$

with the boundary conditions

$$\frac{dU}{d\xi}(\infty) = 0, \quad \frac{d^2U}{d\xi^2}(\infty) = 0,$$

$$\frac{dU}{d\xi}(-\infty) = 0, \quad \frac{d^2U}{d\xi^2}(-\infty) = 0.$$

The moments and forces are then

$$M(\xi) = -\frac{E\delta^2}{24(1-\nu^2)}\,\alpha T_e\,\frac{\delta}{D}\,\frac{d^2U(\xi)}{d\xi^2},$$

$$Q(\xi) = -\frac{E\delta}{24(1-\nu^2)}\,\alpha T_e\left(\frac{\delta}{D}\right)^2\frac{d^3U(\xi)}{d\xi^3},$$

$$N_x(\xi) = -\frac{E\delta}{48(1-\nu^2)}\,\alpha T_e\left(\frac{\delta}{D}\right)^2\frac{d^4U(\xi)}{d\xi^4} =$$

$$= E\delta\alpha T_e[U(\xi) - t(\xi)].$$

Substituting $t(\xi)$ from (11) into (21) and solving this equation, we obtain

$$U(\xi) = \Theta_1 - \frac{4\vartheta_1}{4+\beta_1^4}\exp(-\gamma_1\xi) +$$
$$+ C_1\exp(-k\xi)\cos k\xi + C_2\exp(-k\xi)\sin k\xi, \quad \xi > 0,$$

$$U(\xi) = \Theta_2 + \frac{4\vartheta_2}{4+\beta_2^4}\exp(\gamma_2\xi) +$$
$$+ C_3\exp(k\xi)\cos k\xi + C_4\exp(k\xi)\sin k\xi, \quad \xi < 0,$$

where

$$C_1 = -\frac{1}{2}(\Theta_1-\Theta_2) + \frac{\vartheta_1}{4+\beta_1^4}\left[2+\beta_1-\frac{1}{2}\beta_1^3\right] + \frac{\vartheta_2}{4+\beta_2^4}\left[2-\beta_2+\frac{1}{2}\beta_2^3\right],$$

$$C_2 = -\frac{\vartheta_1}{4+\beta_1^4}\left[\beta_1+\beta_1^2+\frac{1}{2}\beta_1^3\right] + \frac{\vartheta_2}{4+\beta_2^4}\left[\beta_2-\beta_2^2+\frac{1}{2}\beta_2^3\right],$$

$$C_3 = \frac{1}{2}(\Theta_1-\Theta_2) - \frac{\vartheta_1}{4+\beta_1^4}\left[2-\beta_1+\frac{1}{2}\beta_1^3\right] - \frac{\vartheta_2}{4+\beta_2^4}\left[2+\beta_2-\frac{1}{2}\beta_2^3\right],$$

$$C_4 = \frac{\vartheta_1}{4+\beta_1^4}\left[\beta_1-\beta_1^2+\frac{1}{2}\beta_1^3\right] - \frac{\vartheta_2}{4+\beta_2^4}\left[\beta_2+\beta_2^2+\frac{1}{2}\beta_2^3\right],$$

$$\beta_1 = \gamma_1/k, \quad \beta_2 = \gamma_2/k.$$

Thus, thermal radiation can have a significant effect (which increases with the dimensionless parameter σ) on the temperature and thermal-stress distribution in the shell.

Using an analogous method, one can obtain also the approximate solution for the case when the liquid level varies at a constant speed.

NOTATION

T) absolute temperature; x) coordinate; q_{net}) net heat flux lost by the shell by internal radiation; h) heat transfer coefficient; λ) thermal conductivity; ε) emissivity; c_0) Stefan-Boltzmann's constant; D, δ) diameter and thickness of shell; ω) deflection of shell; E) Young's modulus; ν) Poisson's ratio; α) linear coefficient of expansion; M, Q, N_s) moment and forces in the shell.

REFERENCES

1. Yu. A. Surinov, Izv. AN SSR, OTN, no. 7, 1948.
2. A. G. Blokh, Fundamentals of Radiative Heat Transfer [in Russian], Gosnergoizdat, Moscow-Leningrad, 1962.
3. C. M. Usiskin and R. Siegel, Journal of Heat Transfer, Nov., 1960.
4. S. P. Timoshenko and S. Woinwsky-Krieger, Plates and Shells [Russian translation], FM, Moscow, 1963.

16 March 1965 Zhukovskii Central Aerodynamics Institute